ON THE VARIETY OF NETS OF QUADRICS DEFINING TWISTED CUBICS

by

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§1. INTRODUCTION

Fix an algebraically closed field k of characteristic 0, and let V be a vector space over k of dimension 4. Set $\mathbb{P}^3 = \mathbb{P}(V)$, so that $V = \Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ and $S_2(V) = \Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$.

The main objective of this paper is the study of the variety $X \subset \mathrm{Grass}_3(S_2(V))$ consisting of the nets of quadrics generated by the 2-minors of (3×2)-matrices with linear forms as entries. The interest in X stems from the fact that the space of twisted cubic curves may be considered as an open subset of X; in fact, any twisted cubic curve is defined by the vanishing of the 2-minors of a matrix as above.

We shall prove that X is smooth and compact. Hence it gives a natural compactification of the space of twisted cubics. Moreover, it will follow from the construction that X is a minimal compactification in the sense that the complement in X of the space of twisted cubics is an irreducible divisor. Furthermore, we compute - at least in principle - the Chow ring of X by giving algebra generators and relations.

Another compactification of the space of twisted cubics is the Hilbert scheme, or more precisely, the component H of $\mathrm{Hilb}^{3m+1}(\mathbb{P}^3)$ containing the points corresponding to twisted cubics. In [P-S] it is shown that H is smooth. Furthermore, the complement in H of the space of twisted cubics is the union of H_c and H_e, where H_c consists of points corresponding to degenerate twisted cubics that are arithmetically Cohen-Macaulay, and H_e to those that are not, i.e, that consist of a plane, singular cubic curve with an embedded point at a singular point. It is easy to see that all these degenerate curves are contained in exactly three linearly independent quadrics (for the Cohen-Macaulay curves this follows e.g. from [E], for curves with an embedded point this is shown in [P-S]). Hence there is a map $f\colon H \to X \subset \mathrm{Grass}_3(S_2(V))$ which sends a curve to the net of quadrics containing it. Outside H_e, f is an isomorphism, because a curve $C \in H - H_e$ is the intersection of the quadrics in $f(C)$.

If $C \in H_e$, then C is a plane cubic with an embedded point, and $f(C)$ is the net generated by L_0^2, L_0L_1, L_0L_2, where $L_0 = 0$ is an equation of the plane, and $L_0 = L_1 = L_2 = 0$ are equations of the point. Hence $f(H_e)$ is isomorphic to the point-plane incidence correspondence I (which is embedded in $\mathrm{Grass}_3(S_2(V))$ as indicated above).

We strongly believe that $f\colon H \to X$ is the blow-up of X along I. If this is true, we can compute the Chow ring of H. We hope to report on this later.

The restriction of f to $f^{-1}(I)_{red} = H_e \to I$ is isomorphic to a map $\mathbb{P}(N) \to I$, for some rank 7 bundle N on I. Let $g\colon \tilde{\Pi} \to I$ denote the universal plane. Then N is the subbundle of $g_*\mathcal{O}_{\tilde{\Pi}}(3)$

with fiber at (P,Π) consisting of the cubics in Π that are singular at P.

Knowing the Betti numbers of X this suffices to compute the Betti numbers of H (see also [Sch]).

Finally, we remark that because the natural action of a maximal torus in $PGL(V)$ on X and on H has isolated fixed points, the Chow groups are equal to the homology groups, and they are all free abelian [B1,B2].

§2. THE CONSTRUCTION OF X

The points of the compactification X of the space of twisted cubics are nets of quadrics that can be generated by the 2-minors of a (3×2)-matrix with linear forms as entries. We shall now make this connection explicit by exhibiting X as a quotient space.

Let E and F be vector spaces of dimensions 3 and 2 respectively. Set $W = \mathrm{Hom}_k(F, E\otimes V)$. After a choice of bases for E and F we may consider an element $A \in W$ as a matrix (a_{ij}), with $1\leq i\leq 3$, $1\leq j\leq 2$, with entries linear forms, i.e., $a_{ij} \in V$.

For any matrix representation (a_{ij}) of $A\in W$ the maximal minors generate the same subspace E_A of $S_2(V)$. An intrinsic way of constructing E_A is as follows: The map A induces a map $E^\vee \to F^\vee\otimes V$, hence a map $\Lambda^2 E^\vee \to \Lambda^2(F^\vee\otimes V)$. Now there is a canonical map $\Lambda^2(F^\vee\otimes V) \to \Lambda^2 F^\vee\otimes S_2(V)$ and a canonical isomorphism $\Lambda^2 E^\vee \xrightarrow{\approx} E\otimes\Lambda^3 E^\vee$, hence - after identifying the two 1-dimensional vector spaces $\Lambda^3 E^\vee$ and $\Lambda^2 F^\vee$ - we obtain a map $\lambda_A\colon E \to S_2(V)$, whose image is E_A. Note that λ_A is uniquely defined up to a scalar, due to the choice of isomorphism $\Lambda^3 E^\vee \simeq \Lambda^2 F^\vee$.

The group $G_1 = GL(E)\times GL(F)$ acts on W by $(g,h)A = g\otimes \mathrm{id}_V\cdot A\cdot h^{-1}$. Clearly the subgroup $\Gamma = \{(\alpha\cdot \mathrm{id}_E, \alpha\cdot \mathrm{id}_F)\colon \alpha\in k^*\}$ acts trivially on W, hence the group $G = G_1/\Gamma$ acts on W.

For technical reasons we shall consider $P = \mathbb{P}(W)$ and the action of $S = SL(E)\times SL(F)$ on P induced by the action of G on W. If $A \in W$, let $\bar{A} \in P$ denote the corresponding element. Denote by $U \subset W$ the set of maps A such that $\dim E_A = 3$, and denote by $\bar{U} \subset P$ the image of U.

There is a map $\Psi: U \to \mathrm{Grass}_3(S_2(V))$ which sends A to the net E_A. Clearly Ψ factors through X and is G-invariant.

Proposition 1: There exists a projective, smooth geometric quotient $\bar{U}/S$ of $\bar{U}$ by S. The map $\bar{U}/S \to X$ induced by Ψ is an isomorphism and $U \to X$ is a principal homogeneous bundle under G. The rest of this section is devoted to the proof of the above proposition.

Lemma 1: The following statements are equivalent.

(i) $\bar{A}$ is a semistable point under the action of S.

(ii) $\bar{A}$ is a stable point under the action of S.

(iii) $\dim E_A = 3$.

Proof: Assume $\bar{A} \in P$ is not stable. Then there exists an element $(g,h) \in S$ and a 1-parameter subgroup λ of S, on standard form, such that $\mu_\lambda(g \otimes \mathrm{id}_V \cdot \bar{A} \cdot h^{-1}) \leq 0$ (see [N], Prop. 4.11). That λ is on standard form means that

$$\lambda(t) = \left(\begin{pmatrix} t^{\alpha_1} & 0 & 0 \\ 0 & t^{\alpha_2} & 0 \\ 0 & 0 & t^{\alpha_3} \end{pmatrix}, \begin{pmatrix} t^{-\beta_1} & 0 \\ 0 & t^{-\beta_2} \end{pmatrix} \right),$$

where $\alpha_1 \geq \alpha_2 \geq \alpha_3$, $\beta_1 \geq \beta_2$, and $\alpha_1+\alpha_2+\alpha_3 = \beta_1+\beta_2 = 0$.

Clearly $\lambda(t)\cdot A = (t^{\alpha_i+\beta_j} a_{ij})$. If $\alpha_i+\beta_j < 0$, we have $a_{ij} = 0$, because $\bar{A}$ is not stable. Hence $a_{32} = 0$ because $\alpha_3+\beta_2 < 0$. Suppose $\alpha_2+\beta_2 \geq 0$ and $\alpha_3+\beta_1 \geq 0$. Then, adding the two inequalities, we get $-\alpha_1 = \alpha_2+\alpha_3+\beta_1+\beta_2 \geq 0$, which contradicts the fact that α_1 is the largest of three nonzero numbers whose sum is zero. Hence either $a_{22} = 0$ or $a_{31} = 0$, so A is equivalent to a matrix of one of the following types:

$$\begin{pmatrix} * & * \\ * & * \\ 0 & 0 \end{pmatrix} , \begin{pmatrix} * & * \\ * & 0 \\ * & 0 \end{pmatrix} .$$

On the other hand, it is easy to see that matrices of the above types give points of P that are not semistable; in fact, we can use a 1-parameter subgroup with $\alpha_1 = 3$, $\alpha_2 = 2$, $\alpha_3 = -5$, $\beta_1 = 1$, $\beta_2 = -1$, in the first case, and one with $\alpha_1 = 5$, $\alpha_2 = -2$, $\alpha_3 = -3$, $\beta_1 = 4$, $\beta_2 = -4$, in the second case. The following lemma then finishes the proof of Lemma 1.

<u>Lemma 2</u>: Let $A \in W$. Then $\dim E_A \leqslant 2$ if and only if A is equivalent under G to a matrix of one of the above types, i.e., with $a_{32} = a_{31} = 0$ or with $a_{32} = a_{22} = 0$.

<u>Proof</u>: The maximal minors of such matrices are clearly not independent. Hence we may assume that $\dim E_A \leqslant 2$. By performing row operations on A, we may assume that $A=(a_{ij})$, with $a_{21}a_{32}=a_{31}a_{22}$. If $a_{32} = 0$, then either $a_{31} = 0$ or $a_{22} = 0$, and we are done. If $a_{21} = 0$, then either $a_{31} = 0$ or $a_{22} = 0$, and we are done by interchanging the two columns or the last two rows. If all four a_{ij}'s are nonzero, we can write $a_{21} = \gamma\, a_{31}$ and $a_{32} = \gamma^{-1} a_{22}$, or $a_{21} = \gamma a_{22}$ and $a_{32} = \gamma^{-1} a_{31}$, with $\gamma \in k^*$; in both cases an obvious row operation puts A on the desired form. □

We conclude from Lemma 1 and the theory of [M] (see [N], Thm.3.14) that there exists a projective geometric quotient $\bar{U}/S$.

<u>Lemma 3</u>: Let $A \in W$ and let $\mathcal{J}_A \subset \mathcal{O}_P$ denote the sheaf of ideals generated by the quadrics of E_A. Assume $\dim E_A = 3$ and that

$V(\mathfrak{J}_A)$ is not a curve. Then A can be represented by a matrix on the form

$$\begin{pmatrix} 0 & -L_0 \\ L_0 & 0 \\ -L_1 & L_2 \end{pmatrix},$$

where $L_0, L_1, L_2 \in V$ are linearly independent.

<u>Proof</u>: If $V(\mathfrak{J}_A)$ is not a curve, then the 2-minors of A have a common factor which is not a quadric, since otherwise $\dim E_A \leq 1$. Hence they must have a common linear factor. Because $V(\mathfrak{J}_A)$ has no component of codimension greater than 2, the minors are L_0^2, L_0L_1, L_0L_2, for some linearly independent forms L_0, L_1, L_2. Since the relations between these minors obviously are the same as the relations between L_2, L_1, L_0, the columns of A are linear combinations of the columns of the Koszul matrix

$$K = \begin{pmatrix} 0 & L_0 & -L_1 \\ -L_0 & 0 & L_2 \\ L_1 & -L_2 & 0 \end{pmatrix}$$

Hence $A = K\cdot(\alpha_{ij})_{1\leq i\leq 3,\ 1\leq j\leq 2}$, where $\alpha_{ij} \in k$. Working modulo L_1 and L_2 we see that $\det(\alpha_{ij})_{1\leq i\leq 2,\ 1\leq j\leq 2} \neq 0$, and modulo L_0 we obtain $\alpha_{31} = \alpha_{32} = 0$. $\square$

The map $\bar{U}/S \to X$, induced by Ψ, is bijective on closed points. In fact, on points corresponding to nets defining curves, this is clear because of the following. If a net E_A defines a curve, this curve is (a possible degeneration of) a twisted cubic, which in turn determines the matrix A up to the action by G. If E_A does not define a curve, it defines by Lemma 3 a point-plane, and all matrices A defining this point-plane are equivalent under G, again by Lemma 3.

<u>Lemma 4</u>: For any point $A \in U$, the derivative $d_A\Psi$ of Ψ at A has rank 12.

Before proving this lemma we observe that this finishes the proof of Proposition 1. In fact, since the map $\bar{U}/S \to X$ is bijective, the map $U \to X$ has connected fibers. Hence, by Lemma 4, X is smooth, and thus $\bar{U}/S \to X$ is an isomorphism because of Zariski's Main Theorem. Now it is easy to see that $\bar{U}/S$ is a quotient of U by G ($U \to \bar{U}$ is a k^*-bundle, and G is an extension of k^* by S), so we may identify $U/G = \bar{U}/S = X$. To show that $U \to X$ is a principal homogeneous bundle under G, it therefore suffices ([M], 0.9) to check that G acts freely on U. There are two cases to consider. Assume first that $A \in U$ is such that $V(\mathcal{J}_A)$ is a curve. Then there is a resolution of $\mathcal{J}_A$ on $\mathbb{P}^3$,

$$0 \to 2\mathcal{O}(-3) \xrightarrow{A} 3\mathcal{O}(-2) \to \mathcal{J}_A \to 0.$$

Assume $(g,h) \in G_1$ stabilizes A. Then, since $\underline{\mathrm{Hom}}_{\mathbb{P}^3}(\mathcal{J}_A, \mathcal{J}_A) = k^*$, we have

$$\begin{array}{ccccccccc} 0 & \to & 2\mathcal{O}(-3) & \xrightarrow{A} & 3\mathcal{O}(-2) & \to & \mathcal{J}_A & \to & 0 \\ & & \wr\downarrow h & & \wr\downarrow g & & \downarrow\alpha & & \\ 0 & \to & 2\mathcal{O}(-3) & \xrightarrow{A} & 3\mathcal{O}(-2) & \to & \mathcal{J}_A & \to & 0. \end{array}$$

Then $g - \alpha\cdot\mathrm{id}$ induces a map $3\mathcal{O}(-2) \to 2\mathcal{O}(-3)$, which must be zero - hence $g = \alpha\cdot\mathrm{id}$. This implies $h = \alpha\cdot\mathrm{id}$, hence $(g,h) \in \Gamma$, i.e., A has trivial stabilizer in G. In the case $V(\mathcal{J}_A)$ is not a curve, by Lemma 3, A can be represented by a matrix on the form

$$\begin{pmatrix} 0 & -L_0 \\ L_0 & 0 \\ -L_1 & L_2 \end{pmatrix},$$

and one verifies by direct computation that such an A has trivial stabilizer in G.

Proof of Lemma 4: Since $\dim X=12$ and $\dim U = 24$, rank $d_A\Psi = 12$ for general points $A\in U$. There are obvious actions on U and on $\mathrm{Grass}_3(S_2(V))$ by $PGL(V)$, under which Ψ is equivariant. Hence the set of points $A\in U$ where rank $d_A\Psi < 12$ is invariant under the action of $PGL(V)$, and - if nonempty - contains at least one closed orbit. The only closed orbits in U are the orbit consisisting of matrices defining point-planes and that of matrices defining the full second order neighborhood of a line. Therefore we may assume that A is one of the following matrices

$$A_1 = \begin{pmatrix} 0 & -X_0 \\ X_0 & 0 \\ -X_1 & X_2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} X_0 & 0 \\ X_1 & X_0 \\ 0 & X_1 \end{pmatrix}.$$

The tangent space to U at A is just W, and that of

$\mathrm{Grass}_3(S_2(V))$ at $\Psi(A)$ is $\mathrm{Hom}_k(E_A, S_2(V)/E_A)$. A tangent vector τ to U at A is given as $\tau = A + \varepsilon L$, where $L\in W$ and $\varepsilon^2=0$. The map $d_A\Psi(\tau)$ sends a minor of A to the ε-part of the corresponding minor of τ. Hence $d_A\Psi(\tau) = 0$ is equivalent to the three relations

$$(\star) \qquad \begin{vmatrix} a_{i1} & \ell_{i2} \\ a_{j1} & \ell_{j2} \end{vmatrix} + \begin{vmatrix} \ell_{i1} & a_{i2} \\ \ell_{j1} & a_{j2} \end{vmatrix} \in E_A$$

for $1 \leq i < j \leq 3$.

Now $\mathrm{Lie}(G_1) = \mathrm{Lie}(GL(E)) \oplus \mathrm{Lie}(GL(F))$ acts on $\mathrm{Ker}(d_A\Psi)$ via

$$(\mathrm{id}_E+\varepsilon B)(A+\varepsilon L)(\mathrm{id}_F+\varepsilon C) = A + \varepsilon(L+BA+AC)$$

where $B \in \mathrm{Lie}(GL(E))$ and $C \in \mathrm{Lie}(GL(F))$. It will be enough to show that $\mathrm{Ker}(d_A\Psi)$ is the orbit of $A + \varepsilon \cdot 0$, because $\dim \mathrm{Ker}(d_A\Psi) \geqslant 12$ and the orbit is of dimension $\leqslant 12$ since $\mathrm{Lie}(\Gamma) = \{(\gamma \cdot \mathrm{id}_E, -\gamma \cdot \mathrm{id}_R) : \gamma \in k\}$ acts trivially.

We may replace $\tau \in \mathrm{Ker}(d_A\Psi)$ by any other element in the orbit of τ. Hence, if $A=A_1$, we may assume

$$\tau = \begin{pmatrix} 0 & -X_0 \\ X_0 & 0 \\ -X_1 & X_2 \end{pmatrix} + \varepsilon \begin{pmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \\ \ell_{31} & \ell_{32} \end{pmatrix}$$

with $\ell_{11}, \ell_{21}, \ell_{31} \in k[X_2, X_3]$. Using the relations $(\star)$ with $i=2$, $j=3$ and $i=1$, $j=3$, we see that $\ell_{11}=\ell_{21}=0$, and that all $l_{ij} \in k[X_0, X_1, X_2]$. It is now easy to produce an element $(B,C) \in \mathrm{Lie}(G_1)$ such that $(\ell_{ij}) = BA_1 + A_1C$.

If $A=A_2$, then we may assume

$$\tau = \begin{pmatrix} X_0 & 0 \\ X_1 & X_0 \\ 0 & X_1 \end{pmatrix} + \varepsilon \begin{pmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \\ \ell_{31} & \ell_{32} \end{pmatrix},$$

with $\ell_{12}, \ell_{22}, \ell_{32} \in k[X_2, X_3]$. Using the relations $(\star)$ with $i=2$, $j=2$ and $i=1$, $j=3$, we see that $\ell_{12} = \ell_{32} = 0$ and that all $\ell_{ij} \in k[X_0, X_1]$, so that $\ell_{22} = 0$. As above we can then write $(\ell_{ij}) = BA_2 + A_2C$ for some $(B,C) \in \mathrm{Lie}(G_1)$. □

§3. THE CHOW RING OF X

We start by constructing two bundles on X, $\mathcal{E}$ and $\mathcal{F}$, of ranks 3 and 2 respectively, whose Chern classes are algebra generators for the Chow ring $A(X)$ of X. The idea is to try to descend the bundles E_U and F_U on U to X. Clearly G_1 acts on E_U and F_U as follows. If $(g,h) \in G_1$ and $(e,u) \in E_U = E \times U$, then $(g,h)(e,u) = (ge, guh^{-1})$, and similarly on F_U. Since $\Gamma \subset G_1$ does not act trivially on these bundles, we do not get an induced action by $G = G_1/\Gamma$. However, if $\lambda = \mathrm{Hom}_k(\Lambda^3 E, \Lambda^2 F)$, i.e., λ corresponds to the character $(g,h) \to \dfrac{\det h}{\det g}$, it is easily seen that Γ acts trivially on the G_1-bundles $E'_U = E_U \otimes_k \lambda$ and $F'_U = F_U \otimes_k \lambda$. Hence E'_U and F'_U are G-bundles, and because G acts freely on U, these bundles descend to bundles $\mathcal{E}$ and $\mathcal{F}$ on X.

Since by definition $W = \mathrm{Hom}_k(F, E \otimes V)$, there is a universal map $\tilde{A}\colon F_U \to E_U \otimes V$ on U, and hence also a map $\tilde{A} \otimes \mathrm{id}_\lambda\colon F'_U \to E'_U \otimes V$. This map is G-equivariant and descends to a map $\alpha\colon \mathcal{F} \to \mathcal{E} \otimes V$ on X.

<u>Proposition 2</u>: The Chern classes of $\mathcal{E}$ and $\mathcal{F}$ generate the Chow ring $A(X)$ as a $\mathbb{Z}$-algebra.

<u>Proof</u>: The group G_1 is a structure group for the bundle $\mathcal{E} \oplus \mathcal{F}$, so we may construct the principal G_1-homogeneous bundle $\Phi\colon T \to X$ associated with $\mathcal{E} \oplus \mathcal{F}$. Two things should be observed. Firstly, the Chow ring of T is, via Φ^*, isomorphic to $A(X)$ modulo the ideal generated by the Chern classes of $\mathcal{E}$ and $\mathcal{F}$ ([C], Remarques, p. 4-35). Secondly, since $\Phi^*\mathcal{E}$ and $\Phi^*\mathcal{F}$ are trivial, Φ factors through $\Psi\colon U \to X$, and it is easily seen that the induced map $T \to U$ is a

k^*-bundle. Therefore the Chow rings of U and T are isomorphic. Now U is an open subset of the affine space W, and so $A(U) = \mathbf{Z}$. This proves the proposition. □

Remark that $c_1(\mathcal{E}) = c_1(\mathcal{F})$; in fact, as G-bundles, $\Lambda^3 E'_U \cong \Lambda^2 F'_U$, hence we have $\Lambda^3 \mathcal{E} \cong \Lambda^2 \mathcal{F}$. Furthermore, by the definition of Ψ, the restriction to X of the universal subbundle $\mathcal{Q}$ of $S_2(V)$ on $\mathrm{Grass}_3(S_2(V))$ is $\mathcal{E}$.

Let $\pi\colon Y = \mathrm{Grass}_2(Q \otimes V) \to \mathrm{Grass}_3(S_2(V))$ denote the Grassmann bundle of rank 2 subbundles of $\mathcal{Q} \otimes V$, and let $\mathcal{R}$ denote the universal subbundle of $\mathcal{Q} \otimes V$ on Y. The map $\alpha\colon \mathcal{F} \to \mathcal{E} \otimes V$ induces an embedding $i\colon X \to Y$. In fact, α gives $\mathcal{F}$ as a subbundle of $\mathcal{E} \otimes V$ because if α is not injective at a point represented by a (3×2)-matrix A, the two columns of A are linearly dependent, hence all the 2-minors vanish. This is impossible, so we get a map $i\colon X \to Y$, which - being a section over X of the projection $Y \to \mathrm{Grass}_3(S_2(V))$ - is an embedding.

<u>Proposition 3</u>: The class of X in A(Y) is given by

$$[X] = \frac{1}{m}[c(\pi^* Q^{\vee})^{10}(1+c_1(\mathcal{R}) - c_1(\pi^* Q))^{-1}]_{29},$$

where m is some positive integer.

<u>Proof</u>: On Y there are two inclusions, $\pi^* Q \to S_2(V)_Y$ and $\pi^*(Q \otimes \Lambda^3 Q^{-1}) \otimes \Lambda^2\mathcal{R} \to S_2(V)_Y$, the latter being constructed from $\mathcal{R} \to \pi^*\mathcal{Q} \otimes V$ in the same way as we constructed λ_A in §2. The points of X correspond to nets of quadrics that are generated by the 2-minors of a (3×2)-matrix, so it is clear that X consists

of the points of Y where the two above maps of bundles are proportional. Hence X is set-theoretically the scheme Z defined by the 2-minors of the map

$$\mathcal{O}_Y \oplus (\Lambda^2\mathcal{R} \otimes \Lambda^3\pi^*Q^{-1}) \rightarrow \underline{\mathrm{Hom}}_Y(\pi^*Q,\ S_2(V)).$$

Since X has the "right" codimension 29, the class of Z is given by Porteous' formula as $[Z] = [c(\pi^*Q^\vee)^{10}(1+c_1(\mathcal{R})-c_1(\pi^*Q))^{-1}]_{29}$. Since X is irreducible, $[Z] = m[X]$ for some positive integer m. This finishes the proof. (Probably Z is reduced, so that m=1 holds, but we don't need this.) □

<u>Theorem 1</u>: The Chow ring of X is given by $A(X) = A(Y)/\mathfrak{a}$, where

$$\mathfrak{a} = \mathrm{Ann}([c(\pi^*\mathcal{Q}^\vee)^{10}(1+c_1(\mathcal{R})-c_1(\pi^*Q))^{-1}]_{29}).$$

<u>Proof</u>. Recall that $\mathcal{E} = i^*\pi^*\mathcal{Q}$ and $\mathcal{F} = i^*\mathcal{R}$, where $i\colon X \rightarrow Y$ is the embedding. Since A(Y) is generated by the Chern classes of $\pi^*\mathcal{Q}$ and $\mathcal{R}$, and A(X) by those of $\mathcal{E}$ and $\mathcal{F}$ (by Proposition 2), the map $i^*\colon A(Y) \rightarrow A(X)$ is surjective, hence ([S],....), $\mathrm{Ker}(i^*)=\mathrm{Ann}([X])$. The theorem then follows from Proposition 3 and the fact that A(Y) is a free abelian group. □

As a byproduct of the fact that the Chern classes of $\mathcal{E}$ and $\mathcal{F}$ generate A(X) (Proposition 2) and the knowledge of the topological Euler-Poincaré characteristic e(X) of X, we obtain the Betti numbers of X:

i	0	1	2	3	4	5	6
$b_{2i} = b_{2(12-i)}$	1	1	3	4	7	8	10

To see this, we use that $e(X) = 58$ (this will be shown below). Let $R = \oplus R_i$ denote the free, graded $\mathbb{Z}$-algebra with one generator in degree 1, two in degree 2, and one in degree 3. Then $A(X)$ is a quotient of R by a graded ideal $J = \oplus J_i$. The dimensions of the R_i are:

i	0	1	2	3	4	5	6
$\dim R_i$	1	1	3	4	7	9	14

Then $2\sum_{i=0}^{5} \dim R_i + \dim R_6 = 64$. Hence, if $x = \sum_{i=0}^{5} \dim J_i$ and $y = \dim J_6$, we get $2x + y = 6$. Clearly $x=0$, $y=6$ is impossible, since $b_{10} \leqslant b_{12}$. Furthermore, $y=0$ is impossible because if $J_i \neq 0$ for some $i<6$, then $J_6 \neq 0$. Assume $x=y=2$. If $J_4 \neq 0$, then $\dim J_6 \geqslant 3$, so $J_4 = 0$ and $\dim J_5 = 2$. It follows that $J_6 = tJ_5$, where t is the generator of degree 1. The locally split bundle map on X, $\mathcal{E}^\vee \to \mathfrak{F}^\vee \otimes V$, gives a relation of degree 6 between the Chern classes of $\mathcal{E}$ and $\mathfrak{F}$, namely $[c(\mathfrak{F}^\vee)c(\mathcal{E}^\vee)^{-1}]_6 = 0$. This gives an element of J_6 which is not a multiple of t. So the only possibility left is $x=1$, $y=4$, and we are done.

The Euler-Poincaré characteristic of X is computed using the action of a maximal torus of $PGL(V)$ on X. The fixed points are isolated and finite in number, and their number equals $e(X)$ [B1,B2]. If a fixed point of X corresponds to a curve, the support of this curve is contained in the tetrahedron of reference. Hence the curve is either three non coplanar edges (there are 16 such), one edge doubled in a plane (face) union a consecutive edge not contained in that plane (there are 24 such), or the full second order neighborhood of one edge (there are 6 of these). Since the fixed points that do not correspond to curves lie in I, there are $e(I) = 12$ of these. Adding up gives $e(X) = 58$.

Using the fact that the map $f: H \to X$ gives an isomorphism $H - H_e \tilde{\to} X - I$ and that the restriction of f to H_e is a bundle $P(N) \to I$, we obtain the Betti numbers of H.

i	0	1	2	3	4	5	6
$b_{2i} = b_{2(12-i)}$	1	2	6	10	16	19	22

Up to now we have studied determinantal nets of quadrics in $\mathbb{P}^3 = \mathbb{P}(V)$. However, our methods are independent of the dimension of V, and all the proofs carry over to the general case $\dim V = n+1$, with the obvious modifications. Hence we have

<u>Theorem 2</u>: Let V be a vector space of dimension $n+1$. Let $X_n \subset \mathrm{Grass}_3(S_2(V))$ denote the space of determinantal nets of quadrics in $P(V)$. Then X_n is a smooth, projective variety, and its Chow ring is given by

$$A(X_n) = A(Y_n)/\mathfrak{A}_n,$$

where $Y_n = \mathrm{Grass}_2(Q \otimes V)$ and

$$\mathfrak{A}_n = \mathrm{Ann}\left(\left[c(\pi^*Q^\vee)^{\binom{n+2}{2}}(1+c_1(\mathcal{R})-c_1(\pi^*Q))^{-1}\right]_{3\binom{n+1}{2}-1}\right).$$

References

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